DEPTH TWO HOPF SUBALGEBRAS OF SEMISIMPLE HOPF ALGEBRAS

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Abstract. Let H be a finite dimensional semisimple Hopf algebra over an algebraically closed field of characteristic zero. In this note we give a short proof of the fact that a Hopf subalgebra of H is a depth two subalgebra if and only if it is normal Hopf subalgebra.

1. Introduction

The finite depth theory has its roots in the classification of II_1 subfactors. Particularly, depth two theory has been recently extensively investigated by algebraists. This theory is a type of Galois theory for noncommutative rings and has been intensively studied in [3], [5], [6], [4] and other papers. In the theory, the classical Galois group is replaced by a Hopf algebroid. Results from Hopf Galois extensions had also an important influence in the development of this theory.

Hopf algebroids with a separable base algebra are weak Hopf algebras, and they are used in conformal field theory as well as in other subjects. In a recent paper [3], the author used the theory of weak Hopf algebras to prove prove that a depth two Hopf subalgebra of a semisimple Hopf algebra is a normal Hopf subalgebra. This was done by constructing a map from the semisimple Hopf algebra to a weak Hopf algebra. In this note we give a different proof of this result. The character theory for normal Hopf subalgebras developed in [2] and [1] is used.

All algebras and coalgebras in this paper are defined over an algebraically closed ground field k of characteristic zero. For a vector space V by $|V|$ is denoted the dimension $\dim_k V$. The comultiplication, counit and antipode of a Hopf algebra are denoted by Δ, ϵ and S, respectively. For a finite dimensional semisimple Hopf algebra H denote by $\mathrm{Irr}(H)$ the set of irreducible characters of H. All the other notations are those used in [10].

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2. Depth two Hopf subalgebras and normal Hopf subalgebras

A unital subring B is a right depth two subring of A if there is a split epimorphism of $A-B$ bimodules from some A^n onto $A \otimes_B A$ [6]. A similar condition is imposed for left depth two subring. Finitely generated Hopf-Galois extensions satisfy a stronger form of this condition, where the split epimorphism is replaced by an isomorphism of $A-B$ bimodules.

For a separable algebra A with a separable subalgebra B over a filed the following condition is equivalent to the left depth two condition (see Theorem 2.1, item 6 of [4]):

As a natural transformation between functors from the category of B-modules into the category of right A-modules, there is a natural monic from $\mathrm{Ind}_B^A\mathrm{Res}_A^B\mathrm{Ind}_B^A$ into $N\mathrm{Ind}_B^A$ for some positive integer N. In particular, for each pair of simple modules V_B and W_A, the number of isomorphic copies of W,

$$< \mathrm{Ind}_B^A\mathrm{Res}_A^B\mathrm{Ind}_B^A V,\ W > \leq N < \mathrm{Ind}_B^A V,\ W > \tag{2.1}$$

Now suppose $A = H$ is a finite dimensional semisimple Hopf algebra and $B = K$ is a Hopf subalgebra. Since both algebras are semisimple they are also separable.

In terms of character theory the above condition is equivalent to the existence of a positive integer N such that

$$m_H(\alpha \uparrow_K^H \downarrow_K^H \uparrow_K^H,\ \chi) \leq N m_H(\alpha \uparrow_K^H,\ \chi) \tag{2.2}$$

for all irreducible characters $\alpha \in \mathrm{Irr}(K)$ and $\chi \in \mathrm{Irr}(H)$. Since both algebras are semisimple left and right depth two extensions coincide in this situation.

2.1. Equivalence for depth two condition.

It is easy to see that χ is a constituent of $\chi \downarrow_H^K \uparrow_H^K$ for any irreducible H-character $\chi \in \mathrm{Irr}(H)$. Then the depth two condition 2.2 is equivalent to the fact that $\alpha \uparrow_K^H \downarrow_H^K \uparrow_H^K$ and $\alpha \uparrow_H^K$ have the same simple H-constituents for any irreducible character $\alpha \in \mathrm{Irr}(K)$.

2.2. Normal Hopf subalgebras.

Let H be a finite dimensional semisimple Hopf algebra over k. Then H is also cosemisimple [8]. We use the notation $\Lambda_H \in H$ for the idempotent integral of H, ($\epsilon(\Lambda_H) = 1$), and $t_H \in H^*$ for the integral of H^* with $t_H(1) = |H|$. From Proposition 4.1 of [7] it follows that the regular character of H is t_H and therefore

$$t_H = \sum_{\chi \in \mathrm{Irr}(H)} \chi(1)\chi. \tag{2.3}$$

If K is a Hopf subalgebra of H then K is a semisimple and cosemisimple Hopf algebra [10]. A Hopf subalgebra K of H is called normal if $h_1xS(h_2) \in K$ and $S(h_1)xh_2 \in K$ for all $x \in K$ and $h \in H$. If H is semisimple Hopf algebra as above then $S^2 = \mathrm{Id}$ (see [8]) and K is normal in H if and only if $h_1xS(h_2) \in K$ for all $x \in K$ and $h \in H$. If $K^+ = Ker(\epsilon) \cap K$ then K is normal Hopf subalgebra of H if and only if $HK^+ = K^+H$. In this situation $H//K := H/HK^+$ is a quotient Hopf algebra of H via the canonical map $\pi : H \to H//K$ (see Lemma 3.4.2 of [10]). In our settings K is normal in H if and only if $\Lambda_{_K}$ is central in H . For one implication see Lemma 2.16 of [9]. For the other implication, if $\Lambda_{_K}$ is central in H then $HK^+ = H(1 - \Lambda_{_K}) = (1 - \Lambda_{_K})H = K^+H$. For a different proof of this fact see Proposition 2.3. of [3].

2.3. **Restriction to normal Hopf subalgebras.** Let H be a semisimple Hopf algebra over an algebraically closed field k and let K be a normal Hopf subalgebra of H. Define an equivalence relation on the set $\mathrm{Irr}(H)$ by $\chi \sim \mu$ if and only $m_{_K}(\chi \downarrow_K^H,\ \mu \downarrow_K^H) > 0$. This is the equivalence relation $r^{H^*}_{L^*,\ k}$ from [1]. It is proven that $\chi \sim \mu$ if and only if $\frac{\chi\downarrow_K^H}{\chi(1)} = \frac{\mu\downarrow_K^H}{\mu(1)}$ (see Theorem 4.3 of [1]). Thus the restriction of χ and μ to K either have the same irreducible constituents or they don't have common constituents at all.

The above equivalence relation determines an equivalence relation on the set of irreducible characters of K. Two irreducible K-characters α and β are equivalent if and only if they are constituents of $\chi \downarrow_{_K}^H$ for some irreducible character χ of H. Let $\mathcal{C}_1, \cdots \mathcal{C}_s$ be the equivalence classes of the above equivalence relation on $\mathrm{Irr}(H)$. Let $\mathcal{A}_1, \cdots \mathcal{A}_s$ be the corresponding equivalence classes on $\mathrm{Irr}(K)$. The formulae from Section of 4.1 of [1] imply that if $\chi \in \mathcal{C}_i$ then

$$\chi \downarrow_{_K}^H = \frac{\chi(1)}{|\mathcal{A}_i|} \sum_{\alpha \in \mathcal{A}_i} \alpha(1)\alpha \tag{2.4}$$

where $|\mathcal{A}_i| = \sum_{\alpha\in\mathcal{A}_i} \alpha(1)^2$. Also if $\alpha \in \mathcal{A}_i$ then

$$\alpha \uparrow_{_K}^H = \frac{\alpha(1)}{a_i(1)} \frac{|H|}{|K|} \sum_{\chi \in \mathcal{C}_i} \chi(1)\chi \tag{2.5}$$

where $a_i(1) = \sum_{\chi\in\mathcal{C}_i} \chi(1)^2$.

Remark 2.6. These two formulae show that $\alpha \uparrow_{_K}^H \downarrow_{_K}^H \uparrow_{_K}^H$ has the same irreducible K-constituents as $\alpha \uparrow_{_K}^H$ for all irreducible characters $\alpha \in \mathrm{Irr}(K)$. As we have seen in subsection 2.1 this implies that K is a depth two subalgebra of H.

3. Proof of the main result

Let $\epsilon_{_K}$ be the character of the trivial K-module. The following lemma will be used in the proof of the main theorem. This is a slightly weakened version of Corollary 2.5 from [2].

Lemma 3.1. *Let H be a finite dimensional semisimple Hopf algebra and K be a Hopf subalgebra of H. Then K is normal if and only if $\epsilon_{_K} \uparrow_K^H \downarrow_K^H = \frac{|H|}{|K|}\epsilon_{_K}$.*

Proof. Suppose K is normal Hopf subalgebra of H. Then Corollary 2.5 of [2] implies the desired condition.

Conversely, suppose that the above condition is satisfied. By Frobenius reciprocity $m_{_K}(\chi \downarrow_K,\ \epsilon_{_K}) = m_{_H}(\chi,\ \epsilon \uparrow_{_K}^H)$. The above condition implies that for any irreducible character χ of H we have that the value of $m_{_K}(\chi \downarrow_K,\ \epsilon_{_K})$ is either $\chi(1)$ if χ is a constituent of $\epsilon \uparrow_{_K}^H$ or 0 otherwise. But if $\Lambda_{_K}$ is the idempotent integral of K then $m_{_K}(\chi \downarrow_{_K},\ \epsilon_{_K}) = \chi(\Lambda_{_K})$ (see Proposition 1.7.2 of [11]). Thus $\chi(\Lambda_{_K})$ is either zero or $\chi(1)$ for any irreducible character χ of H. This implies that $\Lambda_{_K}$ is a central idempotent of H and therefore K is a normal Hopf subalgebra of H. □

Theorem 3.1. Let H be a finite dimensional semisimple Hopf algebra. A Hopf subalgebra K of H is depth two subalgebra if and only if K is normal in H.

Proof. If K is a normal Hopf algebra then Remark 2.6 shows that K is a depth two subalgebra.

Suppose now that K is a depth two subalgebra of H. Using Frobenius reciprocity the condition 2.2 is equivalent to:

$$m_{_K}(\alpha \uparrow_{_K}^H \downarrow_{_K}^H,\ \chi \downarrow_{_K}^H) \leq N m_{_K}(\alpha,\ \chi \downarrow_{_K}^H) \tag{3.2}$$

for all $\alpha \in \mathrm{Irr}(K)$ and $\chi \in \mathrm{Irr}(H)$. If $\chi = \epsilon_{_H}$ is the trivial H-character then its restriction to K is the trivial K-character and the above condition shows that $m_{_K}(\alpha \uparrow_K^H \downarrow_K^H,\ \epsilon_{_K}) = 0$ if $\alpha \neq \epsilon_{_K}$.

The regular character of K induced to H and then restricted back to K is the regular character of K multiplied $\frac{|H|}{|K|}$. Thus

$$t_{_K} \uparrow_K^H \downarrow_K^H = \frac{|H|}{|K|} t_{_K}.$$

On the other hand using formula 2.3 for K one has

$$t_{_K} \uparrow_K^H \downarrow_K^H = \sum_{\alpha \in \mathrm{Irr}(K)} \alpha(1) \alpha \uparrow_K^H \downarrow_K^H .$$

Thus the multiplicity of $\epsilon_{_K}$ in the above expression is $\frac{|H|}{|K|}$. But in the above sum, $\epsilon_{_K}$ might be constituent only in the term corresponding to the trivial K-character $\alpha = \epsilon_{_K}$ since $m_{_K}(\alpha \uparrow_K^H\downarrow_K^H,\ \epsilon_{_K}) = 0$ for $\alpha \neq \epsilon_{_K}$. Therefore $m_{_K}(\epsilon_{_K} \uparrow_K^H\downarrow_K^H,\ \epsilon_{_K}) = \frac{|H|}{|K|}\epsilon_K$ and a dimension argument implies that $\epsilon_{_K} \uparrow_{_K}^H\downarrow_{_K}^H = \frac{|H|}{|K|}\epsilon_{_K}$. The previous lemma shows that K is normal in H.

$\square$

References

[1] S. Burciu. Coset decomposition for semisimple Hopf algebras. *arXiv:07121719*, 2007.

[2] S. Burciu. Normal Hopf subalgebras semisimple Hopf algebras. *arXiv:0611068*, 2007.

[3] L. Kadison. Semisimple Hopf algebras and their depth two Hopf subalgebras. *arXiv:0807.0599*, 2004.

[4] L. Kadison and B. Kulshammer. Depth two, normality and a trace ideal condition for Frobenius extensions. *arXiv:0409346*, 2004.

[5] L. Kadison and D. Nikshych. Frobenius extensions and weak Hopf algebras. *J. Algebra*, 244:312–342, 2001.

[6] L. Kadison and K. Szlachanyi. Bialgebroid actions on depth two extensions and duality. *Adv. in Math.*, 179:75121, 2003.

[7] R. G. Larson. Characters of Hopf algebras. *J. Algebra*, 117:352–368, 1971.

[8] R.G. Larson and D.E. Radford. Finite dimensional cosemisimple Hopf algebras in characteristic zero are semisimple. *J. Algebra*, 117:267–289, 1988.

[9] A. Masuoka. Self dual Hopf algebras of dimension p^3 obtained by extension. *J. Algebra*, 178(3):791–806, 1995.

[10] S. Montgomery. Hopf algebras and their actions on rings, volume 82. 2nd revised printing, Reg. Conf. Ser. Math, Am. Math. Soc, Providence, 1997.

[11] S. Natale. Semisolvability of semisimple Hopf algebras of low dimension. *Memoirs Amer. Math. Soc.*, 186:123 pp, 2007.

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